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# On Hamilton's principle for surface waves

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The boundary-value problem for irrotational surface waves is derived from a variational integral $I$ with the Lagrangian density $\mathscr{L} = \xi\eta_t - \mathscr{H}$, where $\xi(\mathbf{x}, t)$ is the value of the velocity potential at the free surface, $y = \eta(\mathbf{x}, t)$, and $\mathscr{H}$ is the energy density in $\mathbf{x}$ space. $\mathscr{H}$ then is expressed as a functional of $\xi$ and $\eta$, qua *canonical variables*, by solving a reduced boundary-value problem for the potential, after which the requirement that $I$ be stationary with respect to independent variations of $\xi$ and $\eta$ yields a pair of evolution equations for $\xi$ and $\eta$. The Fourier expansions $\xi = p_n(t)\,\psi_n^*(\mathbf{x})$ and $\eta = q_n(t)\,\psi_n(\mathbf{x})$, where $\{\psi_n\}$ is an orthogonal set of basis functions, reduce $I$ to Hamilton's action integral, in which the complex amplitudes $p_n$ and $q_n$ appear as canonically conjugate co-ordinates, and yield canonical equations for $p_n$ and $q_n$ that are the spectral transforms of the evolution equations for $\xi$ and $\eta$. The evolution equations are reduced (asymptotically) to partial differential equations for $\xi$ and $\eta$ by expanding $\mathscr{H}$ in powers of $\alpha = a/d$ and $\beta = (d/l)^2$, where $a$ and $l$ are scales of amplitude and wavelength. Explicit third approximations are developed for $\beta = O(\alpha)$.

## 1. Introduction

We consider irrotational gravity waves in an ideal homogeneous liquid that fills a rigid basin $B$ (which may be laterally unbounded). Let $\mathbf{x}$ and $y$ be horizontal and vertical co-ordinates, with $y = -d(\mathbf{x})$ at the bottom and $y = 0$ at the quiescent free surface, and let $\phi(\mathbf{x}, y, t)$ be the velocity potential ($\nabla\phi$ = particle velocity), with

$$\phi = \xi(\mathbf{x}, t), \quad y = \eta(\mathbf{x}, t) \tag{1.1a,b}$$

at the displaced free surface. The boundary-value problem may be deduced from the variational principle

$$\delta I = \delta \iint_R \mathscr{L}\, d\mathbf{x}\, dt = 0, \quad \mathscr{L} = \xi\eta_t - \mathscr{H}, \tag{1.2a,b}$$

where

$$\mathscr{H} = \frac{1}{2}\int_{-d}^{\eta} (\nabla\phi)^2\, dy + \tfrac{1}{2}g\eta^2 \equiv \mathscr{T} + \mathscr{V}, \tag{1.2c}$$

$R$ is an arbitrary region in $\mathbf{x}$, $t$ space, $d\mathbf{x}$ is the element of area, and $\mathscr{T}$ and $\mathscr{V}$ are kinetic and potential energy densities in $\mathbf{x}$ space. The requirement that $I$, qua functional of $\phi$ and $\eta$, be stationary with respect to independent variations $\delta\phi$ and $\delta\eta$ that vanish on $\partial R$ (the boundary of $R$) yields

$$\nabla^2\phi = 0 \quad (-d < y < \eta), \tag{1.3}$$

$$\mathbf{n}\,.\,\nabla\phi = 0 \quad \text{on} \quad \partial B \tag{1.4}$$

and
$$\phi_y = \eta_t + \nabla\phi . \nabla\eta, \quad \phi_t + \tfrac{1}{2}(\nabla\phi)^2 + g\eta = 0 \quad (y = \eta), \tag{1.5a,b}$$

where $\partial B$ (the boundary of $B$) comprises both the bottom ($y = -d$) and the lateral boundary (which is distinct from the bottom only if $B$ is at least partially cylindrical).

Surface tension may be incorporated by adding the appropriate terms to $\mathscr{V}$ in (1.2$c$) and to the left-hand side of (1.5$b$).

The variational principle (1.2), as implemented in the penultimate paragraph, is a dynamical extension of the kinematical principle [which follows from Serrin's (1959) statement of Dirichlet's principle]

$$\delta \int_S \left\{ \xi\eta_t - \frac{1}{2}\int_{-d}^{\eta} (\nabla\phi)^2 dy \right\} d\mathbf{x} = 0, \tag{1.6}$$

which yields (1.3)–(1.5$a$) for the variation $\delta\phi$ with $\delta\phi = 0$ on $\partial S$ and $\eta$ fixed. It is a dynamical equivalent of Luke's variational principle (Luke 1967; Whitham 1974, §13.2)

$$\delta \iint_R d\mathbf{x}\, dt \int_{-d}^{\eta} \{\phi_t + \tfrac{1}{2}(\nabla\phi)^2 + gy\}\, dy = 0 \tag{1.7}$$

by virtue of the identities

$$\int_{-d}^{\eta} \phi_t\, dy = \partial_t \int_{-d}^{\eta} \phi\, dy - \xi\eta_t, \quad \int_{-d}^{\eta} gy\, dy = \tfrac{1}{2}g\eta^2 - \tfrac{1}{2}gd^2. \tag{1.8a,b}$$

## 2. Canonical variables

Having established that (1.2) does imply the conventional description of the surface-wave problem, (1.3)–(1.5), in terms of the dependent variables $\phi$ and $\eta$, we henceforth suppose that $I$ has been expressed as a functional of $\xi$, $\eta$ and $\eta_t$ by solving the boundary-value problem posed by (1.3), (1.4) and (1.1) for $\phi$ and substituting the result into (1.2$c$) to obtain the functional

$$\mathscr{L}[\xi, \eta_t] = \xi\eta_t - \mathscr{H}[\xi, \eta], \tag{2.1}$$

where $\mathscr{H}$ is a functional of $\xi$ and $\eta$ that depends implicitly on their values for all $\mathbf{x}$ at each particular value of $t$ (which therefore enters the calculation of $\mathscr{H}$ only as a parameter). The requirement that $I$ be stationary with respect to independent variations $\delta\xi$ and $\delta\eta$ that vanish on $\partial R$ then yields

$$\delta I = \iint_R \left\{ \left(\eta_t - \frac{\delta\mathscr{H}}{\delta\xi}\right)\delta\xi - \left(\xi_t + \frac{\delta\mathscr{H}}{\delta\eta}\right)\delta\eta \right\} d\mathbf{x}\, dt = 0, \tag{2.2}$$

which implies the evolution equations

$$\xi_t = -\frac{\delta\mathscr{H}}{\delta\eta} \equiv -\mathscr{F}_\eta\mathscr{H}, \quad \eta_t = \frac{\delta\mathscr{H}}{\delta\xi} \equiv \mathscr{F}_\xi\mathscr{H}, \tag{2.3a,b}$$

where $\mathscr{F}_\xi\mathscr{H}$ and $\mathscr{F}_\eta\mathscr{H}$ are the functional derivatives of the functional $\mathscr{H}$ with respect to the functions $\xi$ and $\eta$ (see Volterra 1959, §26 ff.; or Finlayson 1972, §9.1).

The Lagrangian functional (2.1) bears a striking resemblance to the Lagrangian $L = \mathbf{p}\dot{\mathbf{q}} - H(\mathbf{p}, \mathbf{q})$ of classical mechanics, wherein the vectors $\mathbf{p} \equiv \{p_n(t)\}$ and $\mathbf{q} \equiv \{q_n(t)\}$ are Hamilton's canonically conjugate (or simply *canonical*) co-ordinates and $H$ is the

Hamiltonian, and the evolution equations (2.3) bear a similar resemblance to Hamilton's canonical equations. Moreover, it follows directly from (2.1) that (note that $\mathscr{L}$ is a local function of $\eta_t$)

$$\xi = \partial \mathscr{L}/\partial \eta_t \tag{2.4}$$

is the analogue of the generalized momentum $\mathbf{p} = \{\partial L/\partial \dot{q}_n\}$. That the analogy is precise may be inferred from a spectral description of the motion, in which the spectral amplitudes of $\xi$ and $\eta$ appear as the canonical co-ordinates and Hamilton's canonical equations appear as the spectral transforms of (2.3); see §3. It therefore seems appropriate to designate $\xi$ and $\eta$ as *canonical variables*, $\mathscr{H}$ as the *Hamiltonian functional* and (2.3) as the *canonical equations* for the surface-wave problem.

We may replace $\mathscr{H}$ in (2.1)–(2.3) by any functional that differs from (1.2$c$) only by a pure divergence in $\mathbf{x}$ space. In particular, a Green's-theorem transformation of the kinetic-energy integral leads to the equivalent functionals

$$\mathscr{H}_* = \mathscr{T}_* + \mathscr{V}, \quad \mathscr{T}_* = \tfrac{1}{2}\xi[\zeta\{1+(\nabla\eta)^2\} - \nabla\xi.\nabla\eta], \tag{2.5$a,b$}$$

where

$$\zeta(\mathbf{x},t) \equiv \zeta[\xi,\eta] \equiv \phi_y(\mathbf{x},\eta,t) \tag{2.6}$$

is determined by the aforementioned solution of (1.3), (1.4) and (1.1).

The evolution equations (2.3) are equivalent to the free-surface conditions (1.5) and may be derived from them in the form

$$\xi_t = -g\eta - \tfrac{1}{2}(\nabla\xi)^2 + \tfrac{1}{2}\zeta^2\{1+(\nabla\eta)^2\} \tag{2.7$a$}$$

and

$$\eta_t = \zeta\{1+(\nabla\eta)^2\} - \nabla\xi.\nabla\eta. \tag{2.7$b$}$$

It therefore is unnecessary to calculate $\mathscr{H}$ explicitly if only the evolution equations are required.†

We remark that (2.5$b$) and (2.7$b$) imply $\xi\eta_t = 2\mathscr{T}_*$ and

$$I = \iint_R (\mathscr{T}_* - \mathscr{V})\, d\mathbf{x}\, dt \equiv \int (T-V)\, dt \equiv \int L\, dt, \tag{2.8}$$

which is Hamilton's action integral. It is possible, in principle, to express $L$ as a functional of $\eta$ and $\eta_t$ by solving (2.7) for $\xi[\eta,\eta_t]$, and then to deduce the governing equation for $\eta$ from Hamilton's principle; however, such a formulation appears to be practical only for a spectral description.

## 3. Spectral description

Pursuing the analogy suggested by (2.1)–(2.4), we pose the Fourier expansions

$$\eta(\mathbf{x},t) = q_n(t)\,\psi_n(\mathbf{x}), \quad \xi(\mathbf{x},t) = p_n(t)\,\psi_n^*(\mathbf{x}), \tag{3.1$a,b$}$$

where $q_n(t)$ and $p_n(t)$ are complex spectral amplitudes, $\{\psi_n(\mathbf{x})\}$ is a complete set of functions in the $\mathbf{x}$ domain $S$ and is orthogonal in the Hermitian sense

$$\langle \psi_m \psi_n^* \rangle = \delta_{mn}, \tag{3.2}$$

† Watson & West (1975) derive equivalents of (2.7) and a quadratic (in powers of $\eta$) approximation to $\zeta$ for deep-water waves ($d = \infty$). They do not use a variational formulation, nor do they mention Hamilton's principle, but they evidently are aware of the canonical character of $\xi$ and $\eta$.

$\langle\ \rangle$ implies an average over $S$, $\psi_n^*$ is the complex conjugate of $\psi_n$, $\delta_{mn}$ is the Kronecker delta, and repeated dummy indices are summed over the spectrum of $\{\psi_n\}$. Substituting (3.1) into (2.1), averaging over $S$, and invoking (3.2), we obtain

$$L \equiv \langle \mathscr{L} \rangle = p_n \dot{q}_n - H, \quad H = \langle \mathscr{H}[p_n \psi_n^*, q_n \psi_n] \rangle. \tag{3.3a, b}$$

The reduced integral $I = \int L\,dt$ is now Hamilton's action integral, and $\delta I = 0$ (Hamilton's principle) implies the canonical equations

$$\dot{p}_n = -\partial H/\partial q_n, \quad \dot{q}_n = \partial H/\partial p_n, \tag{3.4a, b}$$

which are the spectral transforms of the evolution equations (2.3 $a, b$). We remark that $\psi_0 \equiv 1$ is a non-trivial member of $\{\psi_n\}$ in (3.1) and that setting $n = 0$ in (3.4 $a$) yields $\dot{p}_0$, which is required for the calculation of the wave-induced pressure.

The spectral (normal-mode) description is natural for a closed basin, for which the $\psi_n(\mathbf{x})$ are the normal modes and are real. The calculation of both the Lagrangian and the Hamiltonian has been carried out elsewhere (Miles 1976)† and need not be considered further here.

It also may be applied to a domain of large (compared with a characteristic wavelength) lateral extent (cf. Hasselmann 1967; Watson & West 1975). The appropriate eigenfunctions for deep-water waves are

$$\psi_n = e^{i\mathbf{n}\cdot\mathbf{x}}, \quad \psi_n^* = \psi_{-n} \equiv \psi_{\bar{n}}, \tag{3.5a, b}$$

where the subscript $\pm n$ implies a parametric dependence on the two-dimensional wavenumber $\pm\mathbf{n}$ and $n \equiv |\mathbf{n}|$ except in subscripts. The eigenfunctions are orthogonal in the extended sense that

$$\langle \psi_m \cdots \psi_n \rangle = \epsilon_{m\ldots n} = {}^1_0 \quad \text{for} \quad m + \ldots + n \,{}^{=}_{\neq}\, 0, \tag{3.6}$$

where $\psi_m \ldots \psi_n$ is a product of any order (note that $\epsilon_{mn} = \delta_{m\bar{n}}$). The calculation of $H$ follows that for a closed basin (cf. Miles 1976), with the end result

$$H = \tfrac{1}{2}(h_{mn} p_m p_n^* + \delta_{mn} g q_m q_n^*), \tag{3.7}$$

where

$$h_{mn} = \delta_{mn} n + \epsilon_{lm\bar{n}}(\mathbf{m}\,.\,\mathbf{n} - mn)\, q_l + \tfrac{1}{2}\epsilon_{klm\bar{n}}\{(m+n)\,\mathbf{m}\,.\,\mathbf{n} + (\epsilon_{jkm} + \epsilon_{jlm})(jmn + n\mathbf{j}\,.\,\mathbf{m} + m\mathbf{j}\,.\,\mathbf{n})\}\, q_k q_l + \ldots \tag{3.8}$$

is a Hermitian matrix (so that $H$ is a Hermitian form).

## 4. Laterally unbounded domain of constant depth

We pose the solution of (1.3) and (1.4) in a laterally unbounded domain of constant depth in the form

$$\phi(\mathbf{x}, y, t) = \operatorname{sech}(\mathscr{K} d) \cosh\{\mathscr{K}(y+d)\}\, \phi_0(\mathbf{x}, t), \tag{4.1a}$$

where

$$\mathscr{K} \equiv (\partial_y^2 - \nabla^2)^{\frac{1}{2}} \tag{4.1b}$$

† The fact that $\{p_n(t)\}$ is the Fourier transform of $\xi(\mathbf{x}, t)$ for a closed basin, and hence that $\xi$ and $\eta$ are canonical variables, was overlooked in that paper, wherein the Fourier expansion of of $\phi(\mathbf{x}, y, t)$ was related to that of $\eta_t(\mathbf{x}, t)$ through the kinematical principle (1.6) prior to the calculation of the Lagrangian.

is an operational generalization of the scalar wavenumber, and $\phi_0(\mathbf{x},t) \equiv \phi(\mathbf{x},0,t)$. Expanding $\phi$ and $\phi_y$ about $y=0$ and then setting $y=\eta$, we obtain

$$\xi = \sum_{m=0}^{\infty}\left\{\frac{\eta^{2m}}{(2m)!}+\frac{\eta^{2m+1}}{(2m+1)!}\mathscr{M}\right\}\mathscr{K}^{2m}\phi_0, \tag{4.2}$$

and

$$\zeta = \sum_{m=0}^{\infty}\left\{\frac{\eta^{2m+1}}{(2m+1)!}\mathscr{K}^2+\frac{\eta^{2m}}{(2m)!}\mathscr{M}\right\}\mathscr{K}^{2m}\phi_0, \tag{4.3}$$

where

$$\mathscr{M} \equiv \mathscr{K}\tanh\mathscr{K}d \tag{4.4a}$$

$$= \mathscr{K}^2 d\{1-\tfrac{1}{3}(\mathscr{K}d)^2+\tfrac{2}{15}(\mathscr{K}d)^4-\tfrac{17}{315}(\mathscr{K}d)^6+\ldots\}. \tag{4.4b}$$

Inverting the series (4.2) and substituting the result into (4.3), we obtain

$$\phi_0 = \xi-\eta\mathscr{M}\xi+\eta\mathscr{M}\eta\mathscr{M}\xi-\tfrac{1}{2}\eta^2\mathscr{K}^2\xi+O(\eta^3\mathscr{K}^3\xi) \tag{4.5}$$

and

$$\zeta = \mathscr{M}\xi+\eta\mathscr{K}^2\xi-\mathscr{M}\eta\mathscr{M}\xi+(\mathscr{M}\eta\mathscr{M}\eta-\eta\mathscr{K}^2\eta)\mathscr{M}\xi+\tfrac{1}{2}(\eta^2\mathscr{M}-\mathscr{M}\eta^2)\mathscr{K}^2\xi+O(\eta^3\mathscr{K}^4\xi). \tag{4.6}$$

We proceed on the hypothesis that $\mathscr{M}$ may be expanded in even powers of $\mathscr{K}$ according to (4.4$b$). This expansion, in conjunction with the preceding expansion in powers of $\eta$, corresponds to an asymptotic expansino of the solution in powers of the amplitude and dispersion parameters

$$\alpha = a/d, \quad \beta = (d/l)^2, \tag{4.7a, b}$$

where $a$ and $l$ are scales of amplitude and wavelength. We then require only the simple interpretation

$$(\mathscr{K}d)^{2m}f[\xi,\eta] = (-)^m d^{2m}\nabla^{2m}f = O(\beta^m f) \tag{4.8}$$

for any even power of $\mathscr{K}d$ operating on the functional $f$ (which is, by definition, independent of $y$). It follows that $\mathscr{H}$ may be developed in terms of $\xi$, $\eta$, $\nabla\xi$, $\nabla\eta$, $\nabla^2\xi$, $\nabla^2\eta,\ldots$ and that the Frechet operators in (2.3) may be replaced by their Euler–Lagrange expansions,

$$\mathscr{F}_\xi \sim \partial_\xi-\nabla\,.\,(\partial/\partial\nabla\xi)+\nabla^2(\partial^2/\partial\nabla^2\xi)+\ldots \tag{4.9}$$

and similarly for $\mathscr{F}_\eta$. The end result then is to reduce (2.3) to a pair of partial differential equations for $\xi$ and $\eta$.

The preceding development is especially appropriate for the Boussinesq regime, in which nonlinearity and dispersion are both weak but comparably significant, so that $\beta = O(\alpha)$, and the $n$th approximation requires the retention of $O(\alpha^n)$ terms in the evolution equations ($n=1$ corresponds to linear, shallow-water theory and $n=2$ to the conventional Boussinesq approximation). The third approximations to $\zeta$, $\mathscr{H}$ and the evolution equations are given by

$$-\zeta = d\nabla^2\xi+\tfrac{1}{3}d^3\nabla^4\xi+\tfrac{2}{15}d^5\nabla^6\xi+\eta\nabla^2\xi+d^2\nabla^2(\eta\nabla^2\xi), \tag{4.10}$$

$$\mathscr{H} = \tfrac{1}{2}(d+\eta)(\nabla\xi)^2-\tfrac{1}{6}(d^3+3d^2\eta)(\nabla^2\xi)^2+\tfrac{1}{15}d^5(\nabla\nabla^2\xi)^2+\tfrac{1}{2}g\eta^2, \tag{4.11}$$

$$\xi_t+g\eta+\tfrac{1}{2}(\nabla\xi)^2-\tfrac{1}{2}d^2(\nabla^2\xi)^2 = 0 \tag{4.12a}$$

and

$$\eta_t+\nabla\,.\,\{(d+\eta)\nabla\xi\}+\tfrac{1}{3}\nabla^2\{(d^3+3d^2\eta)\nabla^2\xi\}+\tfrac{2}{15}d^5\nabla^6\xi = 0. \tag{4.12b}$$

Equations (4.12$a$, $b$) go slightly beyond, in including the higher-order dispersion term $d^5\nabla^6\xi$, but are otherwise equivalent to, the two-dimensional Boussinesq equations obtained from the variational principle (1.7) by Whitham (1967), his equations (10) and (11) with $h = d+\eta$ and $F = \xi$, who remarked specifically on the simplification that attends the choice of $\xi$ as a dependent variable. They also are equivalent to, but significantly simpler than, results obtained by Benney & Luke (1964), who used $\phi(\mathbf{x}, -d, t)$, rather than $\xi$, as a dependent variable.

The expansion (4.4$b$) is not available for deep-water waves, for which $\mathcal{K}d \to \infty$ and $\mathcal{M} \sim \mathcal{K}$, in consequence of which odd powers of $\mathcal{K}$ enter (4.5) and (4.6), and the appropriate expansion parameter is $a/l \equiv \alpha\beta^{\frac{1}{2}}$. The evolution equations then are not reducible to partial differential equations; however, they may be reduced to integro-(partial) differential equations and appear to be especially well suited for numerical integration (Watson & West 1975).

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